Mathematics 222B Lecture 1 Notes

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1 Introduction to Sobolev Spaces

The main reference for our material on Sobolev spaces will be Ch 5 of Evans' PDE book.

1.1 Sobolev spaces

Definition 1.1. Let $u \in \mathcal{D}'(U)$, where $U \subseteq \mathbb{R}^d$ is open. The **k-th order L^p-based Sobolev norm** of u is

$$\|u\|_{W^{k,p}(U)} := \sum_{\alpha : |\alpha \leq k} \|D^\alpha u\|_{L^p},$$

where we are using the distributional derivative and assume that D^α is an L^p function.

Remark 1.1. Expressions of the form $\|D^\alpha u\|_{L^p}$ arise in the energy method for PDEs with $p = 2$.

Definition 1.2. The **L^p-based Sobolev space of order k on U** is

$$W^{k,p}(U) = \{u \in \mathcal{D}'(U) : \|u\|_{W^{k,p}(U)} < \infty\}.$$

Note that $C_c^\infty(U) \subseteq W^{k,p}(U)$. This allows us to make the following definition:

Definition 1.3. The set of $u \in W^{k,p}(U)$ that vanish (to appropriate orders) on ∂U is

$$W_0^{k,p}(U) = \overline{C_0^\infty(U)}^{\|\cdot\|_{W^{k,p}}}.$$

When $p = 2$, we introduce the notation

$$H^k(U) = W^{k,2}(U), \qquad H_0^k = W_0^{k,2}(U).$$

We can define an inner product on H^k by

$$\langle u, v \rangle_{H^k} = \sum_{\alpha : |\alpha| \leq k} \langle D^\alpha u, D^\alpha v \rangle_{L^2}.$$

Proposition 1.1.

(i) *For all* $k \in \mathbb{Z}_{\geq 0}$ *and* $1 \leq p \leq \infty$, $(W^{k,p}(U), \|\cdot\|_{W^{k,p}})$ *and* $(W_0^{k,p}(U), \|\cdot\|_{W^{k,p}})$ *are Banach spaces.*

(ii) *For all* $k \in \mathbb{Z}_{\geq 0}$, $(H^k(U), \langle \cdot, \cdot \rangle_{H^k})$ *and* $(H_0^k(U), \langle \cdot, \cdot \rangle_{H^k})$ *are Hilbert spaces.*

(iii) *(Fourier-analytic characterization of* H^k*) Given* $u \in H^k(U)$,

$$\begin{aligned} \|u\|_{H^k} &\simeq \|\widehat{u}\|_{L^2} + \||\xi|^k \widehat{u}\|_{L^2} \\ &\simeq \|(1+|\xi|^2)^{k/2}\widehat{u}\|_{L^2}, \end{aligned}$$

where $A \simeq B$ *means* $A \lesssim B$ *and* $B \lesssim A$.

1.2 Duality and Sobolev spaces of negative order

First, we will give a proposition, and then we will explain what is going on.

Proposition 1.2. *For* $k \in \mathbb{Z}_{\geq 0}$ *and* $1 < p < +\infty$,

$$(W_0^{k,p}(U))^* \simeq W^{-k,p'}(U),$$

where $\frac{1}{p'} + \frac{1}{p} = 1$.

Definition 1.4. For $k \in \mathbb{Z}_+$ and $1 < p < +\infty$, the **negative order Sobolev space norm** is

$$\|u\|_{W^{-k,p}(U)} = \inf \left\{ \sum_{\alpha:|\alpha|\leq k} \|g_\alpha\|_{L^p} : u = \sum_{\alpha:|\alpha|\leq k} D^\alpha g_\alpha \right\}.$$

The **negative order Sobolev space** is

$$W^{-k,p}(U) = \{u \in \mathcal{D}'(U) : u = \sum_{\alpha:|\alpha|\leq k} A^\alpha g_\alpha, g_\alpha \in L^p(U)\}.$$

Remark 1.2. If $g \in L^p$, then $D_{x^1} g \in W^{-1,p}(U)$. Compare this with the property of Sobolev spaces that if $u \in W^{k,p}(U)$, then $D_{x^j} u \in W^{k-1,p}(U)$.

Here is the proof of the proposition:

Proof. $(W_0^{k,p}(U))^* \supseteq W^{-k,p'}(U)$: Take $v \in W^{-k,p'}(U)$, so $v = \sum_{\alpha:|\alpha|\leq k} D^\alpha g_\alpha$; we can also take this decomposition so that $\|\sum_{\alpha:|\alpha|\leq k} D^\alpha g_\alpha\|_{L^p} \leq 2\|v\|_{W^{k,p'}(U)}$. Then for $u \in W_0^{k,p}(U)$, we can treat v as a linear functional by

$$\langle v, u \rangle = \int vu \, dx$$

To show that this is bounded,

$$= \sum_{\alpha:|\alpha|\leq k} \int D^\alpha g_\alpha u \, dx$$

First assuming $u \in C_c^\infty$ and then applying a density argument,

$$\begin{aligned} &= \sum_{\alpha:|\alpha|\leq k} \int (-1)^{|\alpha|} g_\alpha D^\alpha u \, dx \\ &\leq \sum_{\alpha:|\alpha|\leq k} |g_\alpha\|_{L^{p'}} \|D^\alpha u\|_{L^p} \\ &\leq C\|v\|_{W^{-k,p'}} \|u\|_{W^{k,p}}. \end{aligned}$$

$(W_0^{k,p}(U))^* \subseteq W^{-k,p'}(U)$: The idea is to use the Hahn-Banach theorem. If X is a nomed vector space and $Y \subseteq X$ with a linear functional $\ell : Y \to \mathbb{R}$ such that $|\ell(u)| \leq C\|u\|$, then there exists an extension $\widetilde{\ell} : X \to \mathbb{R}$ such that $|\widetilde{\ell}(u)| \leq c\|u\|$ and $\widetilde{\ell}|_Y = \ell$.

Let $\ell : W_0^{k,p}(U) \to \mathbb{R}$ be bounded. Define a linear map $C_0^\infty(U) \to L^p(U)^{\oplus K(k)}$ sending $u \mapsto (u, D_{x^1}u, \dots, D_{x^\alpha}u, \dots, D^\alpha u)$, ranging over all multiindices α with $|\alpha| \leq k$. Then $\|Y(u)\| \leq C\|u\|_{W^{k,p}}$, T is injective, and T is an isomorphism of $(C_c^\infty(U), \|\cdot\|_{W^{k,p}})$ with its image $(T(C_c^\infty(U)), \|\cdot\|)$. This gives a bounded map $\widetilde{\ell} : T(C_c^\infty(U)) \to \mathbb{R}$ by $\widetilde{\ell}(Tu) = \ell(u)$. By the Hahn-Banach theorem, $\widetilde{\ell}$ extends to a bounded map $\widetilde{\widetilde{\ell}} : L^p(U)^{\oplus K} \to \mathbb{R}$. That is, $\widetilde{\widetilde{\ell}} \in (L^p(U)^{\oplus K})^* = \{\widetilde{v} = \sum_\alpha \widetilde{g}_\alpha : \widetilde{g}_\alpha \in L^{p'}(U)$. In this picture, for $\widetilde{u} \in L^p(U)^{\oplus K}$, $\langle \widetilde{v}, \widetilde{u}\rangle = \sum_\alpha \langle \widetilde{g}_\alpha, \widetilde{u}_\alpha\rangle$. This means that $\widetilde{\widetilde{\ell}}(\widetilde{v}) = \sum_\alpha \langle \widetilde{g}_\alpha, \widetilde{u}_\alpha\rangle$ for some $\widetilde{g}_\alpha \in L^{p'}(U)$.

This gives

$$\ell(u) = \widetilde{\ell}(Tu) = \widetilde{\widetilde{\ell}}(Tu) = \sum_\alpha \langle \widetilde{g}_\alpha, (Tu)_\alpha\rangle = \sum_\alpha \langle \widetilde{g}_\alpha, D^\alpha u\rangle.$$

Now set $g_\alpha = (-1)^{|\alpha|}\widetilde{g}_\alpha$. □

1.3 Duality in relation to existence and uniqueness

Here is some motivation for our functional analysis. Let X, Y be Banach spaces, and let $P : X \to Y$ be bounded and linear.

- For a given $f \in Y$, does there exists a $u \in X$ such that $Pu = f$? This is the question of existence of a solution to $Pu = f$.
- We can also ask about uniqueness: If $u, u' \in X$ and $Pu = Pu'$, is $u = u'$? That is, if $Pu = 0$, is $u = 0$?

In our course, we usually take P to be a linear differential operator, such as $P = -\Delta$ or $P = \Box$. If we want to solve

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Then we have

$$\int_U |Du|^2 = \int_U -\Delta u \cdot u = \int_U f.$$

This gives

$$\|Du\|_{L^2} \lesssim \left| \int f u \, dx \right|.$$

We are assuming that we have a solution and inferring information about u. This is called an **a priori estimate**.

In energy methods for PDEs, you usually prove a priori estimates, which at first sight, only pertain to uniqueness. However, in fact, a priori estimates are also useful for proving existence because existence vs uniqueness are related to each other using duality. This is the phenomenon in linear algebra where if $A \in \mathbb{R}^{n \times m}$, then A is injective iff A^* is surjective.